Inverses of Multivariate Polynomial Matrices using Discrete Convolution

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***Abstract*— A new method for inversion of rectangular matrices in a multivariate polynomial ring with coefficients in a field is explained. This method requires that the polynomial matrix satisfies the one-to-one mapping criteria defined in [1].**

I. Introduction

The problem of finding inverses of polynomial matrices plays an important role in general theory of linear multi-dimensional systems. For univariate polynomial matrices this problem has been well studied (see [2] and its references). Solutions to multivariate polynomial system of equations is the subject of intensive research and has major applications in the areas of systems and control theory and more recently in the study of multidimensional (m-D) convolutional codes. Significant differences arise in the multivariate case as compared to its univariate counterpart due to the fact that a polynomial ring in more than one variable with coefficients in a field is not a principal ideal domain. Central to the role of finding inverses of multivariate polynomial matrices is the concept of matrix primeness. For $m \geq 2$, new phenomenon arise in the definitions of matrix primeness, and the notions of zero primeness (ZP), minor primeness (MP) and left-factor primeness (LFP) are no longer equivalent (see for example [3]). From the perspective of m-D convolutional codes the primeness properties translate to the existence of parity check matrix or kernel representation and bijective mapping [4], [5], [6]. More specifically, MP $\Leftrightarrow$ Kernel Representation and ZP $\Leftrightarrow$ Invertibility. However, $ZP \Rightarrow MP \Rightarrow LFP$, so the solution of finding an inverse guarantees existence of the primeness hierarchy and will allow for a better understanding of the system dynamics.

The duality between codes and systems is well established [7]. Fornasini and Valcher [5] describe 2-D convolutional codes with finite support using the ring $R[z_1^{\pm 1}, z_2^{\pm 1}]$, where the code is defined as a submodule of the Laurent polynomial ring R^n, and can be viewed as a set of sequences indexed on the discrete plane $\mathbb{Z} \times \mathbb{Z}$. Weiner [8], [4] defines a m-D convolutional code as a submodule of the ring $R[z_1, \ldots, z_m]$ where the code is viewed as a set of sequences indexed on the nonnegative integer lattice $\mathbb{N}^m$. In this paper we seek to exploit the equivalence between polynomial multiplication and discrete convolution to find the inverse of certain multivariate polynomial matrices in the ring $R[z_1, \ldots, z_m]$ that satisfy the one-to-one mapping criteria [1]. We will spend a rather long time establishing notations and definitions in section II so as to elucidate the problem formulation. Section III revisits the one-to-one mapping concept first introduced in [1] for 1-D codes and uses local invertibility to find the inverse of univariate polynomial matrices. We then extend this concept to multivariate polynomial matrices in section IV. Whenever possible examples are provided to illustrate concepts introduced and detailed examples are given in the appendix.

II. Notation and Definitions

A. G is a Linear Transformation

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field with q elements. Let $R = \mathbb{F}[z_1, \ldots, z_m]$ be the polynomial ring in m variables over $\mathbb{F}$. Let $G \in R^{k \times n}$ be an $k \times n$; $k < n$ matrix with elements in R. The map ϕ induces an R-module homomorphism given by matrix multiplication and is defined by

$$\begin{aligned} \phi : \quad R^k & \longrightarrow R^n \\ u & \longmapsto u.G \end{aligned}$$

The $im(\phi)$ is an R-submodule $\mathcal{C}$ of the R-module R^n, and can be viewed as $\mathcal{C} = rowspace(G) = R^k G$. It is well known, see for example [4], [7], that there exists an $\mathbb{F}$-isomorphism between the multivariate polynomial ring R and the m-D finite sequence space S. We explain for completeness.

$$S = \{\omega : \mathbb{N}^m \rightarrow \mathbb{F}\}$$

Where, ω has finite support and elements of $\mathbb{F}$ are attached to the coordinates $(i_1, \ldots i_m)$ of the integer lattice $\mathbb{N}^m$. The isomorphism is given by

$$\begin{aligned} \psi : \quad S & \longrightarrow R \\ \omega & \longmapsto \sum_{(i_1, \ldots i_m)} \omega(i_1, \ldots i_m) \cdot z_1^{i_1} \cdot \ldots \cdot z_m^{i_m} \end{aligned}$$

The coordinates of the lattice $\mathbb{N}^m$ are are associated with monomials of R via the correspondence

$$(i_1, \ldots, i_m) \longleftrightarrow z_1^{i_1} \cdot \ldots \cdot z_m^{i_m}$$

The values $i_1, \ldots, i_m$ form the axes of the m-D sequence space S and we consider the top left point $(i_1, \ldots, i_m)$ of the lattice as the $(0, \ldots, 0)$ coordinate.

Example 1: Consider a 1-D finite sequence $u \in S^2$ with elements from $\mathbb{F}_2$.

$$\begin{array}{ccc} S^2 & & R^2 \\ 11\ \ 01\ \ 10 & \overset{\psi^2}{\longrightarrow} & \begin{bmatrix} 1+z_1^2 \\ 1+z_1 \end{bmatrix} \end{array}$$

Consider a 2-D finite sequence $u \in S^4$ with elements from $\mathbb{F}_2$. The top left point of the lattice is $(i_1, i_2) = (0, 0)$.

$$\begin{array}{ccc} S^4 & & R^4 \\ \begin{array}{cc} 0110 & 1001 \\ 0000 & 0101 \\ 0010 & 1000 \end{array} & \overset{\psi^4}{\longrightarrow} & \begin{bmatrix} z_1 + z_1 z_2^2 \\ 1 + z_1 z_2 \\ 1 + z_2^2 \\ z_1 + z_1 z_2 \end{bmatrix} \end{array}$$

B. Operations in the transform domain S

Discrete convolution in the sequence space domain can be used to compute the coefficients of the product of two polynomials. ie. multiplying two polynomials together, is the same as convolving their coefficients. Let $G \in R^{k \times n}$, with elements $g_x^{(y)}(z) = g_x^{(y)}(z_1, \ldots, z_m) \in R$.

$$G = \begin{bmatrix} g_1^{(1)}(z), & \ldots, & g_1^{(n)}(z) \\ & \vdots & \\ g_k^{(1)}(z), & \ldots, & g_k^{(n)}(z) \end{bmatrix}$$

The sequence space representation of G defined by ψ^{-1} is

$$\begin{pmatrix} g_1^{(1)}, & \ldots, & g_1^{(n)} \\ & \vdots & \\ g_k^{(1)}, & \ldots, & g_k^{(n)} \end{pmatrix}$$

The elements $g_x^{(y)} \in S$ are called *generator sequences* [9].

Let K_{z_i} be the maximum degree along z_i of the polynomial entries in G, that is

$$K_{z_i} = \max_{\substack{1 \le x \le k \\ 1 \le y \le n}} [\deg g_x^{(y)}(z_i)]$$

Equivalently, $K_{z_i} = K_i - 1$, where K_i is the length of the longest generator sequence along the i^{th} axis of the m-D sequence space S.

$$K_i = \max_{\substack{1 \le x \le k \\ 1 \le y \le n}} [\text{len } g_x^{(y)}]$$

For a finite sequence $u \in S^k$ discrete m-D convolution (denoted by $*$) is defined as

$$v = u * g \tag{1}$$

$$\begin{aligned} v^{(1)} &= u^{(1)} * g_1^{(1)} + \cdots + u^{(k)} * g_k^{(1)} \\ &\vdots \\ v^{(n)} &= u^{(1)} * g_1^{(n)} + \cdots + u^{(k)} * g_k^{(n)} \end{aligned}$$

The convolution operation $u^{(x)} * g_x^{(y)}$ implies that for all $(i_1, \ldots, i_m) \ge 0$,

$$v_{(i_1,\ldots,i_m)}^{(y)} = \sum_{l_m=0}^{K_m-1} \cdots \sum_{l_1=0}^{K_1-1} u_{((i_1-l_1),\ldots,(i_m-l_m))}^{(x)} . g_{x(l_1,\ldots,l_m)}^{(y)} \tag{2}$$

where $u_{((i_1-l_1),\ldots,(i_m-l_m))}^{(x)} \triangleq 0$ for all $i_r < l_r$. Addition and multiplication are carried out in modulo-q.

The map φ induces a homomorphism given by discrete convolution and is defined as

$$\begin{aligned} \varphi : \quad S^k &\longrightarrow S^n \\ u &\longmapsto u * g \end{aligned}$$

For the isomorphism $\psi : S \longrightarrow R$, since the notation is discrete convolution in the domain S and polynomial multiplication in the range R, the law of composition translates to

$$\psi(\omega_1 * \omega_2) = \psi(\omega_1).\psi(\omega_2)$$

Example 2: We clarify the above mapping with an example. Let $R = \mathbb{F}_2[z_1]$. Let $G \in R^{2\times 3}$ be given by

$$G = \begin{bmatrix} 1+z_1 & z_1 & 1+z_1 \\ z_1 & 1 & 1 \end{bmatrix}$$

For this univariate polynomial matrix, we have $K_{z_1} = 1$. The sequence space representation of G is

$$\begin{pmatrix} 11 & 01 & 11 \\ 01 & 10 & 10 \end{pmatrix},$$

with $K_1 = 2$. For the sequence $u = 11\ \ 01\ \ 10 \overset{\psi^2}{\rightarrow} [1+z_1^2 \ \ 1+z_1]$, the product

$$v = u.G \quad = \quad [1+z_1^3 \ \ 1+z_1^3 \ \ z_1^2+z_1^3]$$

In the transform domain the equations can be written as $v = u * g$. Since $m = 1$, equation (2) reduces to

$$v_{(i_1)}^{(y)} = \sum_{l_1=0}^{1} u_{(i_1-l_1)}^{(x)} . g_{x(l_1)}^{(y)}$$

$$\begin{aligned} v^{(1)} &= u^{(1)} * g_1^{(1)} + u^{(2)} * g_2^{(1)} \\ &= (101 * 11) + (110 * 01) = 1001 \\ v^{(2)} &= u^{(1)} * g_1^{(2)} + u^{(2)} * g_2^{(2)} \\ &= (101 * 01) + (110 * 10) = 1001 \\ v^{(3)} &= u^{(1)} * g_1^{(3)} + u^{(2)} * g_2^{(3)} \\ &= (101 * 11) + (110 * 10) = 0011 \end{aligned}$$

The input sequence $u \in S^2$ and the output of the convolution $v \in S^3$. The sequence space representation of v is obtained by multiplexing $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$

$$v \quad = \quad 110\ \ 000\ \ 001\ \ 111 \ \overset{\psi^3}{\rightarrow} [1+z_1^3 \quad 1+z_1^3 \quad z_1^2+z_1^3]$$

Example 3: Consider a polynomial vector $u(z_1,z_2) \in R^2,\ R=\mathbb{F}_2[z_1,z_2]$.

$$u = \begin{bmatrix} 1+z_1z_2 & z_2^2 \end{bmatrix}$$

Let $G \in R^{2\times 6}$ be given by

$$G = \begin{bmatrix} z_1^2z_2 & 0 & z_2 & z_1^2 & 1 & 0 \\ 0 & 1+z_1^2z_2 & 0 & z_1z_2 & z_1 & 1+z_1^2+z_2 \end{bmatrix}$$

The product $v=u.G$ give us the transformation $v(z_1,z_2) \in R^6$

$$v = \begin{bmatrix} z_1^2z_2+z_1^3z_2^2 \\ z_2^2+z_1^2z_2^3 \\ z_2+z_1z_2^2 \\ z_1^2+z_1^3z_2+z_1z_2^3 \\ 1+z_1z_2+z_1z_2^2 \\ z_2^2+z_1^2z_2^2+z_2^3 \end{bmatrix}^T$$

For the input sequence $u \in R^2$ we have

$$\begin{array}{ccc} R^2 & & S^2 \\ \begin{bmatrix} 1+z_1z_2 & z_2^2 \end{bmatrix} & \overset{\psi^{-1}}{\rightarrow} & \begin{array}{cc} 10 & 00 \\ 00 & 10 \\ 01 & 00 \end{array} \end{array}$$

$$u^{(1)} = \begin{array}{c} 10 \\ 01 \\ 00 \end{array} \quad u^{(2)} = \begin{array}{c} 00 \\ 00 \\ 10 \end{array}$$

The sequence space representation of G is

$$\left(\begin{array}{cccccc} 000 & 000 & 000 & 001 & 100 & 000 \\ 001 & 000 & 100 & 000 & 000 & 000 \\[4pt] 000 & 100 & 000 & 000 & 010 & 101 \\ 000 & 001 & 000 & 010 & 000 & 100 \end{array} \right),$$

with $K_1=3$ and $K_2=2$. In the transform domain the equations can be written as $v=u*g$. Since $m=2$, equation (2) reduces to

$$v^{(y)}_{(i_1,i_2)} = \sum_{l_2=0}^{1} \sum_{l_1=0}^{2} u^{(x)}_{((i_1-l_1),(i_2-l_2))} . g^{(y)}_{x(l_1,l_2)}$$

$$\begin{array}{ccc} S^2 & & S^6 \\ \begin{array}{c} 10\ 00 \\ 00\ 10 \\ 01\ 00 \end{array} & \overset{*g}{\rightarrow} & \begin{array}{c} 000010\ 000000\ 000100\ 000000 \\ 001000\ 000010\ 100000\ 000100 \\ 010001\ 001010\ 000001\ 100000 \\ 000001\ 000100\ 010000\ 000000 \end{array} \end{array}$$

As per the isomorphism ψ the polynomial representation of the above sequence v is the same as the product $v=u.G$

C. Problem Formulation

The transformation from the ring R to the m-D sequence space S can be visualized with the following commutative diagram

$$\begin{tikzcd} R^k \arrow[r, "\phi", ".G"'] & R^n \arrow[d, "\psi^{n^{-1}}"] \\ S^k \arrow[u, "\psi^k"] & S^n \arrow[l, "*g^{-1}"', "\varphi^{-1}"] \end{tikzcd}$$

Our aim is to find the inverse of the polynomial matrix $G_{k\times n},\ k<n$. G can be viewed as a linear transformation from $R^k \rightarrow R^n$. Throughout section II we have used the isomorphism ψ^{-1} to convert G to its equivalent sequence space representation g. So far we have used g as a transformation from $S^k \rightarrow S^n$ but we do not know the inverse mapping. A solution for g^{-1} in the sequence space S would give us G^{-1} in the ring R. We do this by finding g^{-1} using discrete convolution based on the concept of *local invertibility* and then switch back to the ring R using the isomorphism ψ to give us G^{-1} the $n\times k$ inverse polynomial matrix.

III. Local Invertibility

A. One-to-one Mapping

In [1], [10], [11] Bitzer, Vouk, Dholakia and Koorapaty define a class of 1-D codes known as *Locally Invertible Convolutional Codes* based on the concept of *one-to-one mapping*. From example (2) we see that for $G \in R^{k\times n}$ the convolution operation $v=u*g$ produces n output symbols for every k input symbols. The rate $r=k/n$ is called the *rate of convolution*. The one-to-one mapping technique finds a relationship between the input and output sequence which involves the same number of symbols. We start by explaining this concept for the 1-D case and later extend it to m-dimensions. When $k>1$, the generator sequences are represented in a composite form, where, for a fixed y each $g_x^{(y)}; x=1$ to k is multiplexed into a single sequence, k symbols at a time. The *composite generator sequence* is represented as

$$g = \{\ g^{(1)}, \quad \ldots, \quad g^{(n)}\ \},$$

where each $g^{(y)} \in S^k$. If K_1 is the length of the longest generator sequence $g_x^{(y)}$, then the length of each composite generator sequence $g^{(y)}$ is then simply

$$L_1 = k.K_1 \tag{3}$$

When the generator sequences are expressed in the composite form the convolution $v=u*g$ from equation (1) for the 1-D case is now represented as

$$\begin{aligned} v^{(1)} &= u*g^{(1)} \\ &\vdots \\ v^{(n)} &= u*g^{(n)} \end{aligned}$$

The convolution $u*g^{(y)}$ implies that for all $(i_1)\geq 0$,

$$v^{(y)}_{(i_1)} = \sum_{l_1=1}^{L_1} u_{(i_1-l_1+k)} . \bar{g}^{(y)}_{(L_1-l_1)} \tag{4}$$

where $u_{(i_1-l_1+k)} \triangleq 0$ for all $i_1<l_1$ and $\bar{g}^{(y)}$ is the composite generator sequence reversed k symbols at a time.

Example 4: For the polynomial matrix $G \in R^{2\times 3}$ in example (2) with sequence space representation

$$\left(\begin{array}{ccc} 11 & 01 & 11 \\ 01 & 10 & 10 \end{array} \right),$$

we have $K_1 = 2$. The composite generator sequence is obtained by multiplexing $k = 2$ symbols at a time such that $L_1 = k.K_1 = 4$.

$$g = \{\ 10\,11 \quad 01\,10 \quad 11\,10\ \}$$

g is reversed $k = 2$ symbols at a time to give us

$$\bar{g} = \{\ 11\,10 \quad 10\,01 \quad 10\,11\ \}$$

Since $m = 1$, equation (4) reduces to

$$v^{(y)}_{(i_1)} = \sum_{l_1=1}^{4} u_{(i_1-l_1+2)} \bar{g}^{(y)}_{(4-l_1)}$$

$$\begin{aligned} v^{(1)} &= u * g^{(1)} = 11\ \ 01\ \ 10 * 10\ \ 11 = 1001 \\ v^{(2)} &= u * g^{(2)} = 11\ \ 01\ \ 10 * 01\ \ 10 = 1001 \\ v^{(2)} &= u * g^{(3)} = 11\ \ 01\ \ 10 * 11\ \ 10 = 0011 \end{aligned}$$

The sequence space representation of v is obtained by multiplexing $v^{(1)},\ v^{(2)}$ and $v^{(3)}$

$$v = 110\ \ 000\ \ 001\ \ 111$$

Consider the production of output symbols during the convolution operation.

$Iteration$	$Input$	$Output$
1	L_1	n
2	$L_1 + k$	$2n$
3	$L_1 + 2k$	$3n$
	$\dots$	
i	$L_1 + (i-1)k$	$i.n$

In equation (4), $u_{(i_1-l_1+k)} \triangleq 0$ for all $i_1 < l_1$ implies that the input sequence $u \in S^k$ is padded with zeros. The first L_1 input symbols produce n output symbols. In the next iteration k additional input symbols produce n more output symbols. If we require the number of input and output symbols to be equal we get

$$\begin{aligned} L_1 + (i-1)k &= i.n \\ i &= \frac{L_1 - k}{n - k} \end{aligned}$$

The one-to-one mapping size is then

$$w = i.n = \frac{n(L_1 - k)}{n - k} \tag{5}$$

If the parameters $L_1, n,$ and k are chosen such that w is an positive integer, then, we have a sequence space relationship of w input symbols mapping to w output symbols.

Example 5: For the matrix $G = [1 + z_1 \ \ 1 + z_1 + z_1^2] \overset{\psi^{-1}}{\rightarrow} (110\ \ 111)$, we have $k = 1$, $n = 2$, $K_1 = 3$ and $L_1 = k.K_1 = 3$.

$$g = \{110\ \ 111\}$$

Substituting these values in equation (5) we get the one-to-one mapping size

$$w = \frac{2(3-1)}{2-1} = 4$$

For the matrix $G \in R^{2\times 3}$ from example (2) we have

$$g = \{\ 10\,11 \quad 01\,10 \quad 11\,10\ \}$$

Here $k = 2$, $n = 3$, and $L_1 = 4$. Substituting these values in equation (5) we get the one-to-one mapping size

$$w = \frac{3(4-2)}{3-2} = 6$$

B. Reduced Encoding Matrix

Now consider the map generated by a set E containing w *standard basis* vectors. Let the set $E = \{e_1, \dots, e_w\}$ be defined by

$$\begin{aligned} e_1 &= 100\dots0 \\ e_2 &= 010\dots0 \\ &\dots \\ e_w &= 000\dots1 \end{aligned}$$

where each $e_r \in S^k$ is of size $1\times w$. The convolution operation

$$\hat{g}_r = e_r * g^{(y)};\ y = 1 \text{ to } n \tag{6}$$

gives us an output map, where $\hat{g}_r \in S^n$ is of size $1 \times w$. The $w\times w$ square matrix $\hat{G} = [\hat{g}_1, \dots, \hat{g}_w]^T$, is formed by using each $\hat{g}_r;\ r = 1$ to w, as a row vector.

$$\hat{G}_{w\times w} = \begin{bmatrix} g^{(1)}_{L-1} & \cdots & g^{(n)}_{L-1} & & & & & & & \\ \vdots & & \vdots & & & & & & & \\ g^{(1)}_{L-1-k} & \cdots & g^{(n)}_{L-1-k} & g^{(1)}_{L-1} & \cdots & g^{(n)}_{L-1} & & & & \\ \vdots & & \vdots & \vdots & & \vdots & \ddots & g^{(1)}_{L-1} & \cdots & g^{(n)}_{L-1} \\ g^{(1)}_0 & \cdots & g^{(n)}_0 & \vdots & & \vdots & & \vdots & & \vdots \\ & & & g^{(1)}_0 & \cdots & g^{(n)}_0 & & \vdots & & \vdots \\ & & & & & & \ddots & g^{(1)}_0 & \cdots & g^{(n)}_0 \end{bmatrix}$$

Notice that the first n columns of the reduced encoding matrix are the composite generator sequences reversed k bits at a time. The next n columns are formed by shifting down the composite generator sequences by k rows and so on. This is due to the fact that $\hat{G}$ is a $w\times w$ subsection of the semi-infinite generator matrix [9]. So $\hat{G}$ can be formed by inspection without actually performing the convolution operation defined in equation (6).

Since $\hat{G}$ is constructed from the standard basis input sequence, for any $u \in S^k$, discrete convolution can now be represented by matrix multiplication as shown below by considering $1\times w$ sized subsequences $\hat{u}$ and shifting over the input sequence u, k-symbols at a time.

$$\hat{v} = \hat{u}.\hat{G} \tag{7}$$

If the matrix $\hat{G}$ (called the *reduced encoding matrix* in [1]) is invertible then G is called *Locally Invertible* [10] and the inverse mapping is given by

$$\hat{u} = \hat{v}.\hat{G}^{-1} \tag{8}$$

The sequence u is obtained by considering $1 \times w$ sized subsequences $\hat{v}$ and shifting over the sequence v, n-symbols at a time.

Example 6: For the polynomial matrix $G = [1 + z_1 \;\; 1 + z_1 + z_1^2]$ from example (5) we have $g = \{110 \;\; 111\}$ and one-to-one mapping size $w = 4$. Now equation (6) gives us the transformation

$$\begin{array}{ccccccc} e_1 & & \hat{g}_1 & & e_2 & & \hat{g}_2 \\ 1000 & \overset{*g}{\rightarrow} & 01\ 00 & & 0100 & \overset{*g}{\rightarrow} & 11\ 01 \\ e_3 & & \hat{g}_3 & & e_4 & & \hat{g}_4 \\ 0010 & \overset{*g}{\rightarrow} & 11\ 11 & & 0001 & \overset{*g}{\rightarrow} & 00\ 11 \end{array}$$

The reduced encoding matrix $\hat{G}$ is formed by using each $\hat{g}_r$ as a row vector

$$\hat{G} = \begin{bmatrix} \hat{g}_1 \\ \hat{g}_2 \\ \hat{g}_3 \\ \hat{g}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

For $\hat{u} = 1011 \in S^1$ equations (7) and (8) give us

$$\begin{aligned} \hat{v} &= \hat{u}.\hat{G} = 1011. \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 10 \;\; 00 \\ \hat{u} &= \hat{v}.\hat{G}^{-1} = 10 \;\; 00. \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = 1011 \end{aligned}$$

The sequence $\hat{v} = \hat{u}.G = 10 \;\; 00$ forms only a part of the output sequence $v = u * g = 11 \;\; 11 \;\; \mathit{10} \;\; \mathit{00} \;\; 10 \;\; 01$. The complete output sequence can be obtained by padding $\hat{u}$ with zeros and multiplying $1 \times w$ sized subsequences with $\hat{G}$ by shifting k symbols at a time as follows $\hat{u} = 1011 \simeq 00101100$

$$\begin{array}{lcl} u & & v \\ 0010 & & 1111 \\ \;\;0101 & & \;\;\;\;1110 \\ \;\;\;\;1011 & \xrightarrow{.\hat{G}} & \;\;\;\;\;\;\;\;1000 \\ \;\;\;\;\;\;0110 & & \;\;\;\;\;\;\;\;\;\;\;\;0010 \\ \;\;\;\;\;\;\;\;1100 & & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;1001 \\ 00101100 & \longrightarrow & 11 \;\; 11 \;\; 10 \;\; 00 \;\; 10 \;\; 01 \end{array}$$

Discrete convolution carried out using the operation $\hat{u}.\hat{G}$ leads to a $w - n$ symbol overlap in the output sequence v. This is because $w - k$ symbols of u are reconsidered with every shift of k over the input sequence u.

The inverse mapping is obtained by considering $1 \times w$ sized subsequences of v and multiplying with $\hat{G}^{-1}$ by shifting n symbols at a time.

$$\begin{array}{lcl} v & & u \\ 1111 & & 0010 \\ \;\;\;\;1110 & & \;\;0101 \\ \;\;\;\;\;\;\;\;1000 & \xrightarrow{.\hat{G}^{-1}} & \;\;\;\;1011 \\ \;\;\;\;\;\;\;\;\;\;\;\;0010 & & \;\;\;\;\;\;0110 \\ \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;1001 & & \;\;\;\;\;\;\;\;1100 \\ 11 \;\; 11 \;\; 10 \;\; 00 \;\; 10 \;\; 01 & \longrightarrow & 00101100 \end{array}$$

The $w - k$ symbol overlap in the sequence u is due to the fact that $w - n$ symbols of v are reconsidered with every shift of n during the operation $\hat{u} = \hat{v}.\hat{G}^{-1}$

C. Extracting the Inverse

The commutative diagram shown in section (II-C) can now be viewed using the reduced encoding matrix $\hat{G}$ as follows

$$\begin{tikzcd} R^k \arrow[r, ".G"] & R^n \arrow[d, "\psi^{-1}"] \\ S^k \arrow[u, "\psi"] & S^n \arrow[l, ".\hat{G}^{-1}"] \end{tikzcd}$$

Notice that the transformation from $S^n \rightarrow S^k$ is now carried out using $\hat{G}^{-1}$ instead of g^{-1}. By virtue of the map φ each row of $\hat{G}$ is a finite sequence $\in S^n$. Since $\hat{G}.\hat{G}^{-1} = I$, each column $\hat{g}_r^{-1}$ of $\hat{G}^{-1} = \begin{bmatrix} \hat{g}_1^{-1} & \ldots & \hat{g}_w^{-1} \end{bmatrix}$ is a finite sequence $\in S^n$. From the definition of discrete convolution we see that one of the sequences is reversed. Notice that each row of $\hat{G}$ has elements of g in the reverse order (see example (6)). So the columns of $\hat{G}^{-1}$ will also be finite sequence $\in S^n$ in reverse order. During the operation $\hat{u} = \hat{v}.\hat{G}^{-1}$, observe from the overlapping symbols in the previous example that first $w - k$ columns $\hat{g}_1^{-1}, \ldots, \hat{g}_{w-k}^{-1}$ of $\hat{G}^{-1}$ produce forward shifts along the i^{th} axis in S^k and correspond to multiplication with $z_1^{i_1}$ in the ring R^k. Only the last k columns of $\hat{G}^{-1}$ yield new symbols and therefore represent the inverse generator sequence g^{-1}. These k columns $\hat{g}_{w-k+1}^{-1}, \ldots, \hat{g}_w^{-1}$, correspond to the elements $e_k, \ldots, e_1$ of the standard basis E. Since $\hat{G}^{-1}$ is a linear transformation from $S^n \rightarrow S^k$ and since ψ is an isomorphism from $S \rightarrow R$, the polynomial representation of these columns of $\hat{G}^{-1}$ should give us G^{-1}. ie. the inverse of the polynomial matrix G.

Example 7: From example (6) the row-reduced inverse of the reduced encoding matrix $\hat{G}$ is

$$\hat{G}^{-1} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Each column of $\hat{G}^{-1}$ is a reversed sequence $\in S^2$. Since $k = 1$, the column $\hat{g}_4^{-1}$ corresponds to e_1.

$$\hat{g}_4^{-1} = 1001$$

Reversing $n = 2$ symbols at a time give us inverse composite generator sequence g^{-1}

$$g^{-1} = \{01\ 10\} \xrightarrow{\psi^2} \begin{bmatrix} z_1 \\ 1 \end{bmatrix} = G^{-1}$$

$$\begin{bmatrix} 1+z_1 & 1+z_1+z_1^2 \end{bmatrix} . \begin{bmatrix} z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

As mentioned above the first $w - k$ columns of $\hat{G}^{-1}$ produce forward shifts along the i^{th} axis in S^1 and correspond to multiplication with $z_1^{i_1}$ in the ring R^1.

$$\hat{g}_4^{-1} = 1011$$

Reversing $n = 2$ symbols at a time

$$\{11\ 10\} \xrightarrow{\psi^2} \begin{bmatrix} 1+z_1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+z_1 & 1+z_1+z_1^2 \end{bmatrix} . \begin{bmatrix} 1+z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} z_1 \end{bmatrix}$$

$$\hat{g}_4^{-1} = 0011$$

Reversing $n = 2$ symbols at a time

$$\{11\ 00\} \xrightarrow{\psi^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+z_1 & 1+z_1+z_1^2 \end{bmatrix} . \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} z_1^2 \end{bmatrix}$$

$$\hat{g}_4^{-1} = 1100$$

Reversing $n = 2$ symbols at a time

$$\{00\ 11\} \xrightarrow{\psi^2} \begin{bmatrix} z_1 \\ z_1 \end{bmatrix}$$

$$\begin{bmatrix} 1+z_1 & 1+z_1+z_1^2 \end{bmatrix} . \begin{bmatrix} z_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} z_1^3 \end{bmatrix}$$

For the matrix $G \in R^{2\times 3}$ from example (2) we have $w = 6$ and the row-reduced inverse of the reduced encoding matrix $\hat{G}$ is

$$\hat{G}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The columns $\hat{g}_5^{-1}$ and $\hat{g}_6^{-1}$ corresponding to e_2 and e_1.

$$\hat{g}_5^{-1} = 000011$$
$$\hat{g}_6^{-1} = 011010$$

Reversing $n = 3$ symbols at a time give us the inverse composite generator sequence g^{-1}

$$g^{-1} = \left\{ \begin{array}{c} 011\ 000 \\ 010\ 011 \end{array} \right\} \xrightarrow{\psi^3} \begin{bmatrix} 0 & 0 \\ 1 & 1+z_1 \\ 1 & z_1 \end{bmatrix} = G^{-1}$$

$$\begin{bmatrix} 1+z_1 & z_1 & 1+z_1 \\ z_1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1+z_1 \\ 1 & z_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

IV. Local Invertibility in m-Dimensions

A. Reordering the m-D Sequence Space

The one-to-one mapping size w is a relation between the input and output sequences involving equal number of symbols. For the univariate polynomial ring $R[z_1]$ or the 1-D sequence space this size implicitly translates to a measure of length. Let us examine the parameters involved in computing w as defined in equation (5).

$$w = \frac{n(L_1 - k)}{n - k}$$

Here, L_1 represents the length of the composite generator sequence and k and n represent the number of input and output symbols generated during each step of the convolution operation. The symbols in the sequence are ordered along i_1 since this is the only possible axis in a 1-D space and therefore k and n signify units of length. When extending the concept of one-to-one mapping to m-dimensions or a multivariate polynomial ring $R[z_1, \ldots, z_m]$, it is desirable to maintain this relation along the axis of each dimension. Notice that if we form the composite generator sequence by interleaving, say k symbols along the i_1 axis, then for the term L, we have the lengths $L_1 = k.K_1$ and $L_2 = K_2, \ldots, L_m = K_m$ along each of the m dimensions. But the terms k and n no longer explicitly translate to units of length. The rate $r = k/n$ for m-D convolution means that n output symbols are generated for every k input symbols without specifying their order in the m-D space. S is simply defined as a sequence having finite support with elements (symbols) of $\mathbb{F}$ attached to the coordinates $(i_1, \ldots i_m)$ of the integer lattice $\mathbb{N}^m$. In current literature [8], [4], [12], say for a 2-D sequence space, a sequence S^k is always ordered using k symbols along i_1 and 1 symbol along i_2.

Since S is a linear shift invariant system and operations on S are symbol-wise linearly independent, reordering of symbols attached to each point of the lattice will not change the structure of the sequence as long as the order is maintained during computation.

Proposition 1: When specifying the m-D convolution rate r, the notation

$$r = k/n = k_1/n_1 \times k_2/n_2 \times \cdots \times k_m/n_m \tag{9}$$

specifies the rate of convolution along the dimensions $i_1, \ldots, i_m$ of the m-D sequence space, and the values $k_1, \ldots, k_m$ and $n_1, \ldots, n_m$ define the ordering of the input and output sequence spaces.

When $k > 1$, the m-D composite generator sequence is now formed from the sequence space representation of $G(z_1, \ldots, z_m)$ by interleaving k_i symbols at a time along the i^{th} axis. The composite generator sequence is represented as

$$g = \{\ g^{(1)}, \quad \ldots, \quad g^{(n)}\ \},$$

where each $g^{(y)} \in S^k$. Now, If K_i is the length of the longest generator sequence $g_x^{(y)}$, then the length of each composite generator sequence $g^{(y)}$ is

$$L_i = k_i.K_i \tag{10}$$

For the m-D convolution $v = u * g$, the operation $u * g^{(y)}$ implies that for all $(i_1,\ldots,i_m) \geq 0$

$$v^{(y)}_{(i_1,\ldots,i_m)} = \sum_{l_m=1}^{L_m} \cdots \sum_{l_1=1}^{L_1} u_{((i_1-l_1+k_1),\ldots,(i_m-l_m+k_m))} . \bar{g}^{(y)}_{((L_1-l_1),\ldots,(L_m-l_m))} \tag{11}$$

where $u_{((i_1-l_1+k_1),\ldots,(i_m-l_m+k_m))} \triangleq 0$ for all $i_r < l_r$ and $\bar{g}^y$ is the composite generator sequence reversed $k_1,\ldots,k_m$ symbols at a time along $i_1,\ldots,i_m$ respectively.

Example 8: For the polynomial matrix $G \in R^{2\times 6}$ from example (3) the sequence space representation of G is

$$\begin{pmatrix} \begin{matrix}000\\001\end{matrix} & \begin{matrix}000\\000\end{matrix} & \begin{matrix}000\\100\end{matrix} & \begin{matrix}001\\000\end{matrix} & \begin{matrix}100\\000\end{matrix} & \begin{matrix}000\\000\end{matrix} \\ \begin{matrix}000\\000\end{matrix} & \begin{matrix}100\\001\end{matrix} & \begin{matrix}000\\000\end{matrix} & \begin{matrix}000\\010\end{matrix} & \begin{matrix}010\\000\end{matrix} & \begin{matrix}101\\100\end{matrix} \end{pmatrix}$$

If we specify the rate of convolution as $r = 2/6 = 1/2 \times 2/3$ as defined in proposition (1), then, the composite generator sequence of G is formed by interleaving $k_1 = 1$ symbols along the i_1 and $k_2 = 2$ symbols along the i_2 axes respectively.

$$g = \left\{ \begin{matrix} \begin{matrix}000\\000\end{matrix} & \begin{matrix}000\\100\end{matrix} & \begin{matrix}000\\000\end{matrix} & \begin{matrix}001\\000\end{matrix} & \begin{matrix}100\\010\end{matrix} & \begin{matrix}000\\101\end{matrix} \\ \begin{matrix}001\\000\end{matrix} & \begin{matrix}000\\001\end{matrix} & \begin{matrix}100\\000\end{matrix} & \begin{matrix}000\\010\end{matrix} & \begin{matrix}000\\000\end{matrix} & \begin{matrix}000\\100\end{matrix} \end{matrix} \right\}$$

g is reversed $k_1 = 1$ and $k_2 = 2$ symbols at a time along i_1 and i_2 to form

$$\bar{g} = \left\{ \begin{matrix} \begin{matrix}100\\000\end{matrix} & \begin{matrix}000\\100\end{matrix} & \begin{matrix}001\\000\end{matrix} & \begin{matrix}000\\010\end{matrix} & \begin{matrix}000\\000\end{matrix} & \begin{matrix}000\\001\end{matrix} \\ \begin{matrix}000\\000\end{matrix} & \begin{matrix}000\\001\end{matrix} & \begin{matrix}000\\000\end{matrix} & \begin{matrix}100\\000\end{matrix} & \begin{matrix}001\\010\end{matrix} & \begin{matrix}000\\101\end{matrix} \end{matrix} \right\}$$

The lengths of the composite generator sequence are $L_1 = 3$ and $L_2 = 4$. The rate $r = 1/2 \times 2/3$ imposes an order $k_1 \times k_2 = 1 \times 2$ on the input sequence $u \in S^2$

$$\begin{array}{ccccc} R^2 & & S^2 & & S^2 \\ \begin{bmatrix}1+z_1z_2 & z_2^2\end{bmatrix} & \xrightarrow{\psi^{-1}} & \begin{matrix}10 & 00\\ 00 & 10 \\ 01 & 00\end{matrix} & \rightarrow & \begin{matrix}1\ 0\\0\ 0\\[4pt] 0\ 1\\0\ 0\\[4pt] 0\ 0\\1\ 0\end{matrix} \end{array}$$

2-D convolution $v = u * g$ gives us the transformation $v \in S^6$, where v is order as a $n_1 \times n_2 = 2 \times 3$ sequence as defined by the rate $r = 1/2 \times 2/3$. Since $m = 2$, equation (11) reduces to

$$v^{(y)}_{(i_1,i_2)} = \sum_{l_2=0}^{3} \sum_{l_1=0}^{2} u_{((i_1-l_1+1),(i_2-l_2+2))} \bar{g}^{(y)}_{((3-l_1),(4-l_2))}$$

$$\begin{array}{ccc} S^2 & & S^6 \\ & & \begin{matrix}00 & 00 & 00 & 00\\ 00 & 00 & 01 & 00 \\ 10 & 00 & 00 & 00\end{matrix} \\[4pt] \begin{matrix}1\ 0\\0\ 0\\[4pt] 0\ 1\\0\ 0\\[4pt] 0\ 0\\1\ 0\end{matrix} & \xrightarrow{*\,g} & \begin{matrix}00 & 00 & 10 & 00\\ 10 & 00 & 00 & 01 \\ 00 & 10 & 00 & 00\\[4pt] 01 & 00 & 00 & 10\\ 00 & 10 & 00 & 00 \\ 01 & 10 & 01 & 00\end{matrix} \\[4pt] & & \begin{matrix}00 & 00 & 01 & 00\\ 00 & 01 & 00 & 00 \\ 01 & 00 & 00 & 00\end{matrix} \end{array}$$

As per the isomorphism ψ the polynomial representation of the above sequence v is the same as the product $v = u.G$

B. Reduced Encoding Matrix

The ordering of sequences as per proposition (1) gives us a simple way to extend the one-to-one mapping concept to m-D spaces using

$$w_i = \frac{n_i(L_i - k_i)}{n_i - k_i}; \quad i = 1 \ldots m \tag{12}$$

$$w = w_1 \times \cdots \times w_m \tag{13}$$

w is the m-D one-to-one mapping size, with $w_1 \times \cdots \times w_m$ input symbols mapping to $w_1 \times \cdots \times w_m$ output symbols in the m-D space. For example we have $w_1 \times w_2$ sized rectangles in 2-D, $w_1 \times w_2 \times w_3$ sized cubes in 3-D, $w_1 \times \ldots \times w_m$ sized hypercubes in m-D. As before, each w_i needs to be a positive integer and therefore the rate $r = k/n = k_1/n_1 \times \cdots \times k_m/n_m$ has to be factored such that $n_i > k_i$ in each dimension.

Let E represent the set of w, m-D standard basis vectors.

$$E = \{e_1, \ldots, e_w\}$$

where, $e_r \in S^k$ is a m-D sequence of size $w_1 \times \cdots \times w_m$. Discrete m-D convolution as defined in equation (11) gives us the output map

$$\hat{g}_r = e_r * g^{(y)} \tag{14}$$

where, $\hat{g}_r \in S^n$ is a m-D sequence of size $w_1 \times \cdots \times w_m$. If this multidimensional map is injective, then, the polynomial matrix G is locally invertible. The reduced encoding matrix $\hat{G}$ is constructed by taking each $\hat{g}_r$ and rearranging it as a $1 \times w$, 1-D row vector to form a row of $\hat{G}$.

$$\hat{G} = [\hat{g}_1, \ldots, \hat{g}_w]^T$$

If the $w \times w$ matrix $\hat{G}$ is nonsingular, then, its inverse $\hat{G}^{-1}$ can be found by row-reduction. Since the order of the sequence is preserved during the convolution and row-reduction operations the structure of the sequence is preserved.

For any $u \in S^k$ and $v \in S^n$ the transformations can now be represented by matrix multiplication as

$$\hat{v}_{1\times w} = \hat{u}_{1\times w}.\hat{G}_{w\times w} \tag{15}$$

$$\hat{u}_{1\times w} = \hat{v}_{1\times w}.\hat{G}^{-1}_{w\times w} \tag{16}$$

where $\hat{u}$ and $\hat{v}$ are m-D sequence of size $w_1 \times \cdots \times w_m$ represented as 1-D vectors of size $1 \times w$.

Example 9: For the matrix $G \in R^{2\times 6}$ from example (8) with rate of convolution $r = 2/6 = 1/2 \times 2/3$ we have $k_1 =$

$1, k_2 = 2$ and $n_1 = 1, n_2 = 3$. The lengths of the composite generator sequence are $L_1 = 3$ and $L_2 = 4$. Equation (12) gives us

$$\begin{aligned} w_1 &= \frac{2(3-1)}{2-1} = 4 \\ w_2 &= \frac{3(4-2)}{3-2} = 6 \end{aligned}$$

We now have 4×6 input symbols mapping to 4×6 output symbols and the one-to-one mapping size from equation (13) is $w = 24$.

The set $E = \{e_1, \ldots, e_{24}\}$ of 24 standard basis vectors each of size 4×6 give us the output map

$$\begin{array}{cccccc}
e_1 & & \hat{g}_1 & e_2 & & \hat{g}_2 \\
\begin{array}{c} 1000 \\ 0000 \\[4pt] 0000 \\ 0000 \\[4pt] 0000 \\ 0000 \end{array} & \xrightarrow{*g} & \begin{array}{c} 10\ \ 00 \\ 00\ \ 00 \\ 00\ \ 00 \\[4pt] 00\ \ 00 \\ 00\ \ 00 \\ 00\ \ 00 \end{array} & \begin{array}{c} 0100 \\ 0000 \\[4pt] 0000 \\ 0000 \\[4pt] 0000 \\ 0000 \end{array} & \xrightarrow{*g} & \begin{array}{c} 00\ \ 10 \\ 00\ \ 00 \\ 00\ \ 00 \\[4pt] 00\ \ 00 \\ 00\ \ 00 \\ 00\ \ 00 \end{array} \\
e_5 & & \hat{g}_5 & e_{24} & & \hat{g}_{24} \\
\begin{array}{c} 0000 \\ 1000 \\[4pt] 0000 \\ 0000 \\[4pt] 0000 \\ 0000 \end{array} & \xrightarrow{*g} & \begin{array}{c} 01\ \ 00 \\ 00\ \ 00 \\ 00\ \ 00 \\[4pt] 00\ \ 00 \\ 00\ \ 00 \\ 00\ \ 00 \end{array} & \begin{array}{c} 0000 \\ 0000 \\[4pt] 0000 \\ 0000 \\[4pt] 0000 \\ 0001 \end{array} & \xrightarrow{*g} & \begin{array}{c} 00\ \ 00 \\ 00\ \ 00 \\ 00\ \ 00 \\[4pt] 00\ \ 01 \\ 00\ \ 00 \\ 00\ \ 01 \end{array}
\end{array}$$

Each $\hat{g}_r$ of size 4×6 is represented as a 1×24, 1-D row vector and forms a row of the 24×24 reduced encoding matrix $\hat{G} = [\hat{g}_1, \ldots, \hat{g}_{24}]^T$ (see Appendix (17)). If $\hat{G}$ is nonsingular then the polynomial matrix $G(z_1, z_2)$ is locally invertible.

C. Extracting the Inverse

The output map from equation (14) generates m-D sequences (shown above) of size $w_1 \times \cdots \times w_m$ ordered as $n_1 \times \cdots \times n_m$ as defined in proposition (1). The matrix $\hat{G}$ is constructed by rearranging these m-D sequences as row vectors. Just as in the 1-D case each column of $\hat{G}^{-1}$ is a reversed m-D sequence $\in S^n$ of size $w_1\times\cdots\times w_m$ with order $n_1\times\cdots\times n_m$. The transformations defined in equations (15) and (16) lead to a m-D sequence overlap. Note that in the 1-D case due to the implicit ordering of the input sequence along i_1, only the last k columns of $\hat{G}^{-1}$ yield the inverse. But, for a multivariate polynomial matrix with convolution rate k/n, the order $k_1/n_1 \times\cdots\times k_m/n_m;\ k_i < n_i$ is not unique. For example the composite generator sequences of a rate $2/6$ 2-D polynomial matrix could be ordered as $1/2\times 2/3$ or $2/3\times 1/2$ which in turn would impose an order of 1×2 or 2×1 on the input sequence respectively. Therefore the columns of $\hat{G}^{-1}$ that represent the inverse generator sequence g^{-1} depends on the numbering of the $w_1\times\cdots\times w_m$ sized standard basis E. For our 2-D example we have chosen to number the standard basis in a row wise incrementing order

$$\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ e_5 & e_6 & e_7 & e_8 \\[4pt] e_9 & e_{10} & e_{11} & e_{12} \\ e_{13} & e_{14} & e_{15} & e_{16} \\[4pt] e_{17} & e_{18} & e_{19} & e_{20} \\ e_{21} & e_{22} & e_{23} & e_{24} \end{array}$$

The columns of $\hat{G}^{-1}$ corresponding to the elements of the standard basis that match the order $k_1\times\cdots\times k_m$ will yield the polynomial inverse.

Example 10: Each column of $\hat{G}^{-1}_{24\times 24}$ (appendix (18)) is a reversed sequence $\in S^6$ ordered as a $n_1\times n_2 = 2\times 3$, column vector. The elements e_1 and e_5 match the order $k_1\times k_2 = 1\times 2$. The columns of $\hat{G}^{-1}_{24\times 24}$ that correspond to e_5 and e_1 are

$$\begin{aligned} \hat{g}^{-1}_{20} &= 000000010110011000100010 \\ \hat{g}^{-1}_{24} &= 000000001000000110000000 \end{aligned}$$

Rearranging as a 4×6, 2-D sequence gives us

$$\begin{array}{cc} 0000 & 0000 \\ 0001 & 0000 \\ 0110 & 1000 \\ 0110 & 0001 \\ 0010 & 1000 \\ 0010 & 0000 \end{array}$$

Reversing $n_1 = 2$ and $n_2 = 3$ bits along i_1 and i_2

$$\begin{array}{c} 0000 \\ 0001 \\ 0110 \\ 0110 \\ 0010 \\ 0010 \end{array} \to \begin{array}{c} 00\ \ 00 \\ 01\ \ 00 \\ 10\ \ 01 \\[4pt] 10\ \ 01 \\ 10\ \ 00 \\ 10\ \ 00 \end{array} \to \begin{array}{c} 10\ \ 01 \\ 10\ \ 00 \\ 10\ \ 00 \\[4pt] 00\ \ 00 \\ 01\ \ 00 \\ 10\ \ 01 \end{array} \xrightarrow{\psi^6} \begin{bmatrix} 1 \\ z_1 \\ 1 \\ z_2 \\ 1+z_2 \\ z_1 z_2 \end{bmatrix}$$

$$\begin{array}{c} 0000 \\ 0000 \\ 1000 \\ 0001 \\ 1000 \\ 0000 \end{array} \to \begin{array}{c} 00\ \ 00 \\ 00\ \ 00 \\ 00\ \ 10 \\[4pt] 01\ \ 00 \\ 00\ \ 10 \\ 00\ \ 00 \end{array} \to \begin{array}{c} 01\ \ 00 \\ 00\ \ 10 \\ 00\ \ 00 \\[4pt] 00\ \ 00 \\ 00\ \ 00 \\ 00\ \ 10 \end{array} \xrightarrow{\psi^6} \begin{bmatrix} 0 \\ 1 \\ z_1 \\ 0 \\ z_1 z_2 \\ 0 \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} 1 & 0 \\ z_1 & 1 \\ 1 & z_1 \\ z_2 & 0 \\ 1+z_2 & z_1 z_2 \\ z_1 z_2 & 0 \end{bmatrix}$$

D. Diversity of m-D Sequence Ordering

The matrix $G_{k\times n}; k < n$ is rectangular, so unlike the case of a nonsingular matrix, which has a single unique inverse, G may not have an inverse or it may have a multiplicity of *generalized inverses* [13], [14]. From proposition (1) we know that the order $k_1/n_1 \times\cdots\times k_m/n_m;\ k_i < n_i$ is not unique, so it is natural to question the choice of the convolution rate $r = 2/6 = 1/2\times 2/3$ in the above example. It is interesting to note what happens if the rate $r = 2/6 = 2/3\times 1/2$ is chosen while ordering the composite generator sequence. The sequence space representation of G is

$$\begin{pmatrix} 000 & 000 & 000 & 001 & 100 & 000 \\ 001 & 000 & 100 & 000 & 000 & 000 \\[4pt] 000 & 100 & 000 & 000 & 010 & 101 \\ 000 & 001 & 000 & 010 & 000 & 100 \end{pmatrix}$$

If we specify the rate of convolution as $r = 2/6 = 2/3\times 1/2$, then, the composite generator sequence of g is formed by interleaving $k_1 = 2$ symbols along the i_1 and $k_2 = 1$ symbols along the i_2 axes respectively.

$$\left\{ \begin{matrix} 00\,00\,00 & 01\,00\,00 & 00\,00\,00 & 00\,00\,10 & 10\,01\,00 & 01\,00\,01 \\ 00\,00\,10 & 00\,00\,01 & 10\,00\,00 & 00\,01\,00 & 00\,00\,00 & 01\,00\,00 \end{matrix} \right\}$$

The lengths of the composite generator sequence are $L_1 = 6$ and $L_2 = 2$. Equation (12) gives us $w_1 = 12$, $w_2 = 2$ and $w = w_1 \times w_2 = 24$. The reduced encoding matrix $\hat{G}_{24\times 24}$ (see appendix (19)) is constructed from the 2-D standard basis E with each e_i of size 12×2. Notice that rows 9 and 11 of $\hat{G}$ are all-zero rows and hence make it singular. Therefore the polynomial inverse G^{-1} cannot be be found using this method for the rate $r = 2/6 = 2/3 \times 1/2$. However, for some G, the reduced encoding matrix $\hat{G}$ is nonsingular when factored using different rates (appendix B) and the polynomial inverse G^{-1} obtained for each rate need not be unique.

APPENDIX

A. *Rate* $r = 2/6 = 1/2 \times 2/3$

The reduced encoding matrix $\hat{G} = [\hat{g}_1, \ldots, \hat{g}_{24}]^T$ constructed from the 2-D standard basis E is

$$\hat{G}_{24\times 24} = \begin{bmatrix} 100000000000000000000000 \\ 001000000000000000000000 \\ 000010000000000000000000 \\ 000000100000000000000000 \\ 010000000000000000000000 \\ 000101000000000000000000 \\ 000000010100000000000000 \\ 000000000001000000000000 \\ 000001000000100000000000 \\ 000000010000000100000000 \\ 000000001000000010000000 \\ 000000000010000000100000 \\ 000000000100010000000000 \\ 000000001001000101000000 \\ 010000000110000000010100 \\ 000100000001000000000001 \\ 000000000000000001000000 \\ 000000000000000000010000 \\ 000000000000000000001000 \\ 000000000000000000000010 \\ 000000000000000000000100 \\ 000000000000000000001001 \\ 000000000000010000000110 \\ 000000000000000100000001 \end{bmatrix} \quad (17)$$

$$\hat{G}^{-1}_{24\times 24} = \begin{bmatrix} 100000000000000000000000 \\ 000010000000000000000000 \\ 010000000000000000000000 \\ 000000010000000100100100 \\ 001000000000000000000000 \\ 000001010000000100100100 \\ 000100000000000000000000 \\ 000000100000100000011010 \\ 000000010000010010100101 \\ 000000000000100000011010 \\ 000010000000101001010010 \\ 000000010000000000000000 \\ 000001011000000100100100 \\ 000000000000000000011010 \\ 000000100100100000011010 \\ 000000000000000000100101 \\ 000000010010010010100101 \\ 000000000000000010000000 \\ 000010000001101001010010 \\ 000000000000000001000000 \\ 000000000000000000100000 \\ 000000000000000000001000 \\ 000000000000000000010000 \\ 000000000000000000100100 \end{bmatrix} \quad (18)$$

If we specify the rate of convolution as $r = 2/6 = 2/3 \times 1/2$, then, the reduced encoding matrix constructed from the 2-D standard basis E is

$$\hat{G}_{24\times 24} = \begin{bmatrix} 100000000000000000000000 \\ 010000000000000000000000 \\ 000100000000000000000000 \\ 000010000000100000000000 \\ 000000100000000000000000 \\ 000000010000001100000000 \\ 000000000100000000000000 \\ 000000000010000001100000 \\ 000000000000000000000000 \\ 000000000000000000001100 \\ 000000000000000000000000 \\ 000000000000000000000001 \\ 000000000000100000000000 \\ 000000000000001000000000 \\ 000000000000000100000000 \\ 000000000000010001000000 \\ 001000000000010000100000 \\ 010000000000010100001000 \\ 000001000000000010000100 \\ 000010000000000001010001 \\ 000000001000000000010000 \\ 000000010000000000001010 \\ 000000000001000000000010 \\ 000000000010000000000001 \end{bmatrix} \quad (19)$$

B. *Non-Unique Inverses*

Consider the $G(z_1, z_2) \in R^{2\times 6}$ polynomial matrix

$$G = \begin{bmatrix} z_1 z_2^2 & 1 \\ z_1 z_2 & z_2 + z_1 z_2^2 \\ 1 + z_2^2 + z_1 z_2^2 & z_2^2 \\ z_2 + z_2^2 & 1 \\ z_2^2 & z_1 + z_2 \\ 1 + z_1 + z_1 z_2 & 1 + z_1 z_2^2 \end{bmatrix}^T$$

The sequence space representation of G is

$$\begin{pmatrix} \begin{matrix} 00 & 00 & 10 & 00 & 00 & 11 \\ 00 & 01 & 00 & 10 & 00 & 01 \\ 01 & 00 & 11 & 10 & 10 & 00 \end{matrix} \\ \begin{matrix} 10 & 00 & 00 & 10 & 01 & 10 \\ 00 & 10 & 00 & 00 & 10 & 00 \\ 00 & 01 & 10 & 00 & 00 & 01 \end{matrix} \end{pmatrix}$$

If we specify the rate of convolution as $r = 2/6 = 2/3 \times 1/2$ as defined in proposition (1), then, the composite generator sequence of G is formed by interleaving $k_1 = 2$ symbols along the i_1 and $k_2 = 1$ symbols along the i_2 axes respectively.

$$g = \left\{ \begin{matrix} 01\,00 & 00\,00 & 10\,00 & 01\,00 & 00\,01 & 11\,10 \\ 00\,00 & 01\,10 & 00\,00 & 10\,00 & 01\,00 & 00\,10 \\ 00\,10 & 00\,01 & 11\,10 & 10\,00 & 10\,00 & 00\,01 \end{matrix} \right\}$$

The lengths of the composite generator sequence are $L_1 = 4$ and $L_2 = 3$. The rate $r = 2/6 = 2/3 \times 1/2$ gives us $k_1 = 2, k_2 = 1$ and $n_1 = 3, n_2 = 2$.

$$\begin{aligned} w_1 &= \frac{3(4-2)}{3-2} = 6 \\ w_2 &= \frac{2(3-1)}{2-1} = 4 \\ w &= w_1 \times w_2 = 24 \end{aligned}$$

The reduced encoding matrix $\hat{G} = [\hat{g}_1, \dots, \hat{g}_{24}]^T$ constructed from the 2-D standard basis E is

$$\hat{G}_{24\times 24} = \begin{bmatrix} 101000000000000000000000 \\ 010000001000000000000000 \\ 001101110000000000000000 \\ 001010000001000000000000 \\ 000001000110000000000000 \\ 000001000000000000000000 \\ 010000001000101000000000 \\ 000000000000010000001000 \\ 000010100001001101110000 \\ 010000010000001010000001 \\ 000000000100000001000110 \\ 000010000001000000100000 \\ 000000001000010000001000 \\ 000000010000000000000000 \\ 001000000100100001010001 \\ 100000101010010000010000 \\ 000001000000100000000100 \\ 000100000010100001000010 \\ 000000000000000000001000 \\ 000000000000000000010000 \\ 000000000000001000001001 \\ 000000000000100000101010 \\ 000000000000000001000001 \\ 000000000000000100000101 \end{bmatrix} \tag{20}$$

$$\hat{G}^{-1}_{24\times 24} = \begin{bmatrix} 011010110000011101111100 \\ 010000001000010000000000 \\ 111010110000011101111100 \\ 100110111100010111101111 \\ 111000010111000111011000 \\ 000001000000000000000000 \\ 010101001100011010010011 \\ 000000000000010000000000 \\ 000000010000100000000000 \\ 001100111100100011011111 \\ 001111111100100011011111 \\ 000110100111011010100100 \\ 100111001010100100101101 \\ 000000010000000000100000 \\ 110111101010100100101101 \\ 110000001101111100100000 \\ 010000010100110000101000 \\ 110111101010100100000111 \\ 011011111001111011110101 \\ 000000000000000000010000 \\ 000000000000000000100000 \\ 000111100111011000100100 \\ 111100110011011111111100 \\ 110111101010100100000101 \end{bmatrix} \tag{21}$$

Each column of $\hat{G}^{-1}_{24\times 24}$ is a reversed sequence $\in S^6$ ordered as a $n_1 \times n_2 = 3\times 2$, column vector. The elements e_1 and e_2 match the order $k_1 \times k_2 = 2\times 1$. The columns of $\hat{G}^{-1}_{24\times 24}$ that correspond to e_2 and e_1 are

$$\begin{aligned} \hat{g}^{-1}_{23} &= 000100100110000001000000 \\ \hat{g}^{-1}_{24} &= 000100100110101001100001 \end{aligned}$$

Rearranging as a 6×4 subsequence gives us

$$\begin{array}{cc} 000100 & 000100 \\ 100110 & 100110 \\ 000001 & 101001 \\ 000000 & 100001 \end{array}$$

Reversing $n_1 = 3$ and $n_2 = 2$ bits along z_1 and z_2

$$\begin{array}{c} 000100 \\ 100110 \\ 000001 \\ 000000 \end{array} \rightarrow \begin{array}{cc} 100 & 000 \\ 110 & 100 \\ 001 & 000 \\ 000 & 000 \end{array} \rightarrow \begin{array}{cc} 001 & 000 \\ 000 & 000 \\ 100 & 000 \\ 110 & 100 \end{array} \xrightarrow{\psi^6} \begin{bmatrix} z_2 \\ 0 \\ 1 \\ z_2 + z_1 z_2 \\ z_2 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} 000100 \\ 100110 \\ 101001 \\ 100001 \end{array} \rightarrow \begin{array}{cc} 100 & 000 \\ 110 & 100 \\ 001 & 101 \\ 001 & 100 \end{array} \rightarrow \begin{array}{cc} 001 & 101 \\ 001 & 100 \\ 100 & 000 \\ 110 & 100 \end{array} \xrightarrow{\psi^6} \begin{bmatrix} z_1 + z_2 \\ 0 \\ 1 + z_1 \\ z_1 + z_2 + z_1 z_2 \\ z_2 \\ 1 \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} z_2 & z_1 + z_2 \\ 0 & 0 \\ 1 & 1 + z_1 \\ z_2 + z_1 z_2 & z_1 + z_2 + z_1 z_2 \\ z_2 & z_2 \\ 0 & 1 \end{bmatrix}$$

If we specify the rate of convolution as $r = 2/6 = 1/2\times\, 2/3$ as defined in proposition (1), then, the composite generator sequence of G is formed by interleaving $k_1 = 1$ symbols along the i_1 and $k_2 = 2$ symbols along the i_2 axes respectively.

$$g \;=\; \left\{ \begin{array}{cccccc} 00 & 00 & 10 & 00 & 00 & 11 \\ 10 & 00 & 00 & 10 & 01 & 10 \\ 00 & 01 & 00 & 10 & 00 & 01 \\ 00 & 10 & 00 & 00 & 10 & 00 \\ 01 & 00 & 11 & 10 & 10 & 00 \\ 00 & 01 & 10 & 00 & 00 & 01 \end{array} \right\}$$

The lengths of the composite generator sequence are $L_1 = 2$ and $L_2 = 6$. The rate $r = 2/6 = 1/2\times\, 2/3$ gives us $k_1 = 1, k_2 = 2$ and $n_1 = 2, n_2 = 3$.

$$\begin{aligned} w_1 &= \frac{2(2-1)}{2-1} = 2 \\ w_2 &= \frac{3(6-2)}{3-2} = 12 \\ w &= w_1 \times w_2 = 24 \end{aligned}$$

The reduced encoding matrix $\hat{G} = [\hat{g}_1, \ldots, \hat{g}_{24}]^T$ constructed from the 2-D standard basis E is

$$\hat{G}_{24\times 24} = \begin{bmatrix} 101000000000000000000000 \\ 001110000000000000000000 \\ 010001000000000000000000 \\ 001000000000000000000000 \\ 010001101000000000000000 \\ 000100001110000000000000 \\ 000000010001000000000000 \\ 010010001000000000000000 \\ 000001010001101000000000 \\ 001001000100001110000000 \\ 000010000000010001000000 \\ 100101010010001000000000 \\ 000000000001010001101000 \\ 000000001001000100001110 \\ 000000000010000000010001 \\ 000000100101010010001000 \\ 000000000000000001010001 \\ 000000000000001001000100 \\ 000000000000000010000000 \\ 000000000000100101010010 \\ 000000000000000000000001 \\ 000000000000000000001001 \\ 000000000000000000000010 \\ 000000000000000000100101 \end{bmatrix} \tag{22}$$

$$\hat{G}^{-1}_{24\times 24} = \begin{bmatrix} 100100000000000000000000 \\ 111000100011101011000101 \\ 000100000000000000000000 \\ 101010100101111110000111 \\ 111110100101111110000111 \\ 110000100011101011000101 \\ 001100010110010101000010 \\ 011111100010001110101100 \\ 000110010110010101000010 \\ 000100101100000010111010 \\ 101001011111101001111111 \\ 011111000010001110101100 \\ 111000110001011001010100 \\ 010111111000011101111000 \\ 001000111010110010010001 \\ 111000110001011011001110 \\ 000000000000000000100000 \\ 101001011111100011111111 \\ 100001100101010000100111 \\ 101001011111100001110111 \\ 000000000000000000001100 \\ 100001100101010000101110 \\ 000000000000000000000010 \\ 000000000000000000001000 \end{bmatrix} \tag{23}$$

Each column of $\hat{G}^{-1}_{24\times 24}$ is a reversed sequence $\in S^6$ ordered as a $n_1 \times n_2 = 2\times 3$, column vector. The elements e_1 and e_3 match the order $k_1 \times k_2 = 1\times 2$. The columns of $\hat{G}^{-1}_{24\times 24}$ that correspond to e_3 and e_1 are

$$\begin{aligned} \hat{g}^{-1}_{22} &= 010111010011100101111100 \\ \hat{g}^{-1}_{24} &= 010111000010001001110000 \end{aligned}$$

Rearranging as a 2×12 subsequence gives us

$$\begin{array}{cc} 01 & 01 \\ 01 & 01 \\ 11 & 11 \\ 01 & 00 \\ 00 & 00 \\ 11 & 10 \\ 10 & 00 \\ 01 & 10 \\ 01 & 01 \\ 11 & 11 \\ 11 & 00 \\ 00 & 00 \end{array}$$

Reversing $n_1 = 2$ and $n_2 = 3$ bits along z_1 and z_2

$$\begin{array}{c} 01 \\ 01 \\ 11 \\ 01 \\ 00 \\ 11 \\ 10 \\ 01 \\ 01 \\ 11 \\ 11 \\ 00 \end{array} \rightarrow \begin{array}{c} 01 \\ 01 \\ 11 \\[4pt] 01 \\ 00 \\ 11 \\[4pt] 10 \\ 01 \\ 01 \\[4pt] 11 \\ 11 \\ 00 \end{array} \rightarrow \begin{array}{c} 11 \\ 11 \\ 00 \\[4pt] 10 \\ 01 \\ 01 \\[4pt] 01 \\ 00 \\ 11 \\[4pt] 01 \\ 01 \\ 11 \end{array} \xrightarrow{\psi^6} \begin{bmatrix} 1 + z_2 \\ 1 + z_2^2 + z_2^3 \\ 1 \\ 1 + z_2 + z_2^3 \\ z_2^2 + z_2^3 \\ z_2 + z_2^2 + z_2^3 \end{bmatrix}$$

$$\begin{array}{c} 01 \\ 01 \\ 11 \\ 00 \\ 00 \\ 10 \\ 00 \\ 10 \\ 01 \\ 11 \\ 00 \\ 00 \end{array} \rightarrow \begin{array}{c} 01 \\ 01 \\ 11 \\[4pt] 00 \\ 00 \\ 10 \\[4pt] 00 \\ 10 \\ 01 \\[4pt] 11 \\ 00 \\ 00 \end{array} \rightarrow \begin{array}{c} 11 \\ 00 \\ 00 \\[4pt] 00 \\ 10 \\ 01 \\[4pt] 00 \\ 00 \\ 10 \\[4pt] 01 \\ 01 \\ 11 \end{array} \xrightarrow{\psi^6} \begin{bmatrix} 1 \\ 1 + z_2^3 \\ z_2 \\ z_2^3 \\ z_2^2 + z_2^3 \\ z_2 + z_2^3 \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} 1 + z_2 & 1 \\ 1 + z_2^2 + z_2^3 & 1 + z_2^3 \\ 1 & z_2 \\ 1 + z_2 + z_2^3 & z_2^3 \\ z_2^2 + z_2^3 & z_2^2 + z_2^3 \\ z_2 + z_2^2 + z_2^3 & z_2 + z_2^3 \end{bmatrix}$$

C. Rate $r = 1/4 = 1/2\times 1/2$

Consider the $G(z_1, z_2) \in R^{1\times 4}$ polynomial matrix

$$G = \begin{bmatrix} 1 + z_1^2 z_2 + z_2^2 + z_1^2 z_2^2 \\ 1 + z_1 z_2^2 \\ 1 + z_1^2 + z_1 z_2 \\ 1 + z_1^2 z_2^2 \end{bmatrix}^T$$

The sequence space representation of G is

$$\begin{pmatrix} \begin{matrix}100\\001\\101\end{matrix} & \begin{matrix}100\\000\\010\end{matrix} & \begin{matrix}101\\010\\000\end{matrix} & \begin{matrix}100\\000\\001\end{matrix} \end{pmatrix}$$

Since $k = 1$ the sequences are not interleaved. If we specify the rate of convolution as $r = 1/4 = 1/2\times 1/2$ as defined in proposition (1), then, the composite generator sequence of G is and is the same as the sequence space representation above.

$$g = \left\{ \begin{matrix}100\\001\\101\end{matrix} \quad \begin{matrix}100\\000\\010\end{matrix} \quad \begin{matrix}101\\010\\000\end{matrix} \quad \begin{matrix}100\\000\\001\end{matrix} \right\}$$

The lengths of the composite generator sequence are $L_1 = 3$ and $L_2 = 3$. The rate $r = 1/2\times 1/2$ gives us $k_1 = 1, k_2 = 1$ and $n_1 = 2, n_2 = 2$.

$$\begin{aligned} w_1 &= \frac{2(3-1)}{2-1} = 4 \\ w_2 &= \frac{2(3-1)}{2-1} = 4 \\ w &= w_1 \times w_2 = 16 \end{aligned}$$

The reduced encoding matrix $\hat{G} = [\hat{g}_1, \ldots, \hat{g}_{16}]^T$ constructed from the 2-D standard basis E is

$$\hat{G}_{16\times 16} = \begin{bmatrix} 1000010000000000 \\ 0110000100000000 \\ 1001000000000000 \\ 0010000000000000 \\ 1000000010000100 \\ 0010100001100001 \\ 0000001010010000 \\ 0000000000100000 \\ 0000100010000000 \\ 0000001000101000 \\ 1100110000000010 \\ 0011001100000000 \\ 0000000000001000 \\ 0000000000000010 \\ 0000000011001100 \\ 0000000000110011 \end{bmatrix} \tag{24}$$

$$\hat{G}^{-1}_{16\times 16} = \begin{bmatrix} 0001111111000111 \\ 0111111010011111 \\ 0001000000000000 \\ 0011111111000111 \\ 1111111010111011 \\ 1001111111000111 \\ 0000000101001000 \\ 0010111010011111 \\ 1111111000111011 \\ 0001011111001101 \\ 0000000100000000 \\ 1111110101110011 \\ 0000000000001000 \\ 1110100111111100 \\ 0000000000000100 \\ 1111110001110110 \end{bmatrix} \tag{25}$$

Each column of $\hat{G}^{-1}_{16\times 16}$ is a reversed sequence $\in S^4$ ordered as a $n_1 \times n_2 = 2\times 2$, column vector. The element e_1 matches the order $k_1 \times k_2 = 1\times 1$. The column of $\hat{G}^{-1}_{16\times 16}$ that corresponds to e_1 is

$$\hat{g}^{-1}_{16} = 1101110111010000$$

Rearranging as a 4×4 subsequence gives us

$$\begin{matrix} 1101 \\ 1101 \\ 1101 \\ 0000 \end{matrix}$$

Reversing $n_1 = 2$ and $n_2 = 2$ bits along z_1 and z_2

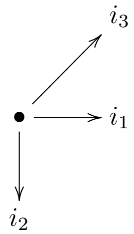
$$\begin{matrix} 1101 \\ 1101 \\ 1101 \\ 0000 \end{matrix} \rightarrow \begin{matrix} 01 \;\; 11 \\ 01 \;\; 11 \\ \\ 01 \;\; 11 \\ 00 \;\; 00 \end{matrix} \rightarrow \begin{matrix} 01 \;\; 11 \\ 00 \;\; 00 \\ \\ 01 \;\; 11 \\ 01 \;\; 11 \end{matrix} \xrightarrow{\psi^4} \begin{bmatrix} z_1 + z_1 z_2 \\ 1 + z_1 + z_2 + z_1 z_2 \\ z_1 z_2 \\ z_2 + z_1 z_2 \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} z_1 + z_1 z_2 \\ 1 + z_1 + z_2 + z_1 z_2 \\ z_1 z_2 \\ z_2 + z_1 z_2 \end{bmatrix}$$

D. 3-D Generator Sequences

Consider the $G(z_1, z_2, z_3) \in R^{1\times 8}$ polynomial matrix

$$G = \begin{bmatrix} z_1 + z_2 + z_1 z_2 z_3 \\ z_3 + z_2 z_3 \\ 1 + z_1 z_2 + z_1 z_3 \\ z_2 + z_1 z_3 \\ z_1 z_2 + z_1 z_3 \\ z_1 + z_1 z_2 + z_1 z_3 \\ z_2 + z_1 z_3 + z_1 z_2 z_3 \\ z_1 z_2 + z_1 z_3 + z_2 z_3 \end{bmatrix}^T$$



Since $k=1$ the sequences are not interleaved. If we specify the rate of convolution as $r=1/4=1/2\times 1/2\times 1/2$ as defined in proposition (1), then, the composite generator sequence of G is

$$g=\left\{\begin{array}{cccccccc}01&00&10&00&00&01&00&00\\10&00&01&10&01&01&10&01\end{array}\right\};\;\; i_3=0$$

$$\left\{\begin{array}{cccccccc}00&10&01&01&01&01&01&01\\01&10&00&00&00&00&01&10\end{array}\right\};\;\; i_3=1$$

The lengths of the composite generator sequence are $L_1=2$, $L_2=2$ and $L_2=2$. The rate $r=1/2\times 1/2\times 1/2$ gives us

$$\begin{aligned} w_1 &= \frac{2(2-1)}{2-1}=2\\ w_2 &= \frac{2(2-1)}{2-1}=2\\ w_3 &= \frac{2(2-1)}{2-1}=2\\ w &= w_1\times w_2\times w_3=8\end{aligned}$$

The reduced encoding matrix $\hat{G}=[\hat{g}_1,\ldots,\hat{g}_8]^T$ constructed from the 3-D standard basis E is

$$\hat{G}_{8\times 8}=\begin{bmatrix}10000010\\01000001\\00111111\\01000000\\00101101\\10010010\\10000100\\00100000\end{bmatrix} \tag{26}$$

$$\hat{G}^{-1}_{8\times 8}=\begin{bmatrix}00101100\\00010000\\00000001\\10000100\\01110111\\00101110\\10101100\\01010000\end{bmatrix} \tag{27}$$

Each column of $\hat{G}^{-1}_{8\times 8}$ is a reversed sequence $\in S^8$ ordered as a $n_1\times n_2\times n_3=2\times 2\times 2$, column vector. The element e_1 matches the order $k_1\times k_2\times k_3=1\times 1\times 1$. The column of $\hat{G}^{-1}_{8\times 8}$ that corresponds to e_1 is

$$\hat{g}_8^{-1}=00101000$$

Rearranging as a $2\times 2\times 2$ subsequence gives us

$$\begin{array}{ccc}00 & & 10\\ 10 & ; & 00\\ i_3=0 & & i_3=1\end{array}$$

Reversing $n_1=2$, $n_2=2$ and $n_3=2$ bits along i_1, i_2 and i_3

$$\begin{array}{c}00\\10\end{array};\begin{array}{c}10\\00\end{array}\overset{\psi^8}{\rightarrow}\begin{bmatrix}0&0&1&0&1&0&0&0\end{bmatrix}^T$$

$$G^{-1}=\begin{bmatrix}0&0&1&0&1&0&0&0\end{bmatrix}^T$$

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